The low-rate approximation to $\mu(\rho',\tau')$ in the source-pulser method

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For the source-pulser method of measuring dead times Müller has derived the appropriate interval densities [1] and the corresponding formula for computing the dead time from observed counting rates [2], viz.,

$$\tau = \frac{1+\mu}{2\mu r_\nu}\left[1 - \sqrt{1 - \frac{4\mu}{(1+\mu)^2}\cdot\frac{r_\nu}{r}\cdot\frac{r+\nu-r_\nu}{\nu}}\right] \quad --- \quad (1)$$

This is to be compared to the well-known approximate formula,

$$\tau_o = \frac{1}{r}\left[1 - \sqrt{\frac{r_\nu - r}{\nu}}\right] \quad -------------- \quad (2)$$

which has been shown to break down for high pulser duty cycles, i.e. when $\nu \geq 1/3\tau$ [3]. The notation here and in the following is the same as Müller's [1,2].

In equation (1) the factor μ which has been tabulated in [1] as a function of ρ' and τ' ($\rho' = \rho/\nu$, $\tau' = \nu\tau$) has no known analytical form and must be evaluated from the interval densities by an iterative procedure [1].

The remarkable agreement between experimental observations [3], calculations and Monte Carlo simulations reported in [1] and [2] leaves no doubt as to the correctness of Müller's solution to this long standing problem.

The purpose of this note is to give explicitly μ_o, the approximate expression for μ which makes equation (1) equivalent to equation (2); i.e. for which $\tau = \tau_o$. From (2) we get

$$r_\nu = r + \nu\,(1 - r\tau_o)^2 .$$

Since $r = \rho/(1+\rho\tau)$ and $1 - r\tau = 1/(1+\rho\tau)$ and because we are interested in the region where τ_o is a good approximation to τ, we can write $r \approx \rho/(1+\rho\tau_o)$ and $1 - r\tau_o \approx 1/(1+\rho\tau_o)$ to get

$$r_\nu \approx \frac{\rho(1+\rho\tau_o) + \nu}{(1+\rho\tau_o)^2} \quad ---------- \quad (3)$$

From [2] (equation (12) with K'=1)

$$r_\nu = \frac{\rho(1-\mu\nu\tau)+\nu}{1+(1-\mu\nu\tau)\rho\tau} \quad \text{--------} \quad (4)$$

To solve for μ_o, (4) is equated to (3) with μ and τ replaced by μ_o and τ_o; i.e.,

$$\frac{\rho(1+\rho\tau_o)+\nu}{(1+\rho\tau_o)^2} = \frac{\rho(1-\mu_o\nu\tau_o)+\nu}{1+(1-\mu_o\nu\tau_o)\rho\tau_o} ,$$

from which

$$\mu_o = \frac{1+\rho\tau_o}{1+\rho\tau_o-\nu\tau_o} .$$

Equivalent expressions in the notation of [1] are

$$\mu_o(\rho',\tau_o') = \frac{1+\rho'\tau_o'}{1+(\rho'-1)\tau_o'} \text{ or } 1 + \frac{\tau_o'}{1+(\rho'-1)\tau_o'} \quad \text{----} \quad (5)$$

As expected, a comparison of $\mu_o(\rho',\tau_o')$ with $\mu(\rho',\tau')$ given in Table 2 of [1] shows close agreement at rates for which the memory effect of the pulser series is negligible and τ_o is a good approximation to τ. Table 1 lists a few examples.

Since $\tau_o/\tau = 1$ for $\mu = \mu_o$ and $\left(\frac{\tau_o}{\tau} - 1\right) \approx \frac{1}{2}\left(\frac{\mu}{\mu_o} - 1\right)$, the difference between μ and μ_o can be estimated from Figure 1, a reproduction of Figure 2a from [2]. The oscillatory behaviour of μ for $\tau' < 0.5$ is clearer from the figure than from Table 2 of [1]; e.g., in the table $\mu-\mu_o \approx 10^{-4}$ for $(\rho',\tau') = (1,0.30)$ as well as (1,0.25) and the user might assume this would be true for intermediate values whereas from the figure it can be deduced that $\mu-\mu_o \approx 10^{-3}$ at (1,0.28).

Because μ_o corresponds to τ_o and equation (2) it is of no help in computing dead times; where μ_o is a good enough approximation, equation (2) may be used. Otherwise equation (1) and better values of μ, i.e., the tabulated ones, are needed. For more accurate comparisons, however, μ_o could be the starting point for a finer-mesh tabulation of $\mu'(\rho',\tau')$ where $\mu' = \mu-\mu_o$. Such a tabulation would be useful only if a form of equation (1) as a function of μ' could be found that would be reasonably convenient for desk calculation. Three-figure values of μ' would be adequate for most purposes.

It is interesting to note that putting $\mu=1$ reduces equation (1) to the classical result for two random sources while, as shown above, putting $\mu=\mu_o$ reduces equation (1) to the usual equation for one random and one periodic source. Whether or not there is an expression for μ which reduces equation (1) to the simple but exact solution for two periodic sources has yet to be deomonstrated. If not, it would be of interest to know if equation (1) can be rewritten in terms of a more general parameter that will accommodate all three cases.

References

[1] Müller, J.W., Rapport BIPM-76/3 (1976)

[2] " " Rapport BIPM-76/5 (1976)

[3] Baerg, A.P., NRC report PXNR-2414 (1972)

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TABLE 1

Some values of $\mu_o(\rho',\tau')$. Values of $\mu(\rho',\tau')$ from [1] are given in parentheses for comparison

	ρ' = 0.2	0.5	1	2	5
τ' = 0.1	1.10870 (1.1087)	1.10526 (1.1053)	1.10000 (1.1000)	1.09091 (1.0909)	1.07143 (1.0714)
0.2	1.23810 (1.2381)	1.22222 (1.2222)	1.20000 (1.2000)	1.16667 (1.1667)	1.11111 (1.1113)
0.3	1.39474 (1.3946)	1.35294 (1.3525)	1.30000 (1.3001)	1.23077 (1.2361)	1.13636 (1.1474)
0.4	1.58824 (1.5992)	1.50000 (1.5135)	1.40000 (1.4030)	1.28571 (1.2620)	1.15385 (1.1148)
0.5	1.83333 (1.9064)	1.66667 (1.7870)	1.50000 (1.6321)	1.33333 (1.4323)	1.16667 (1.1987)

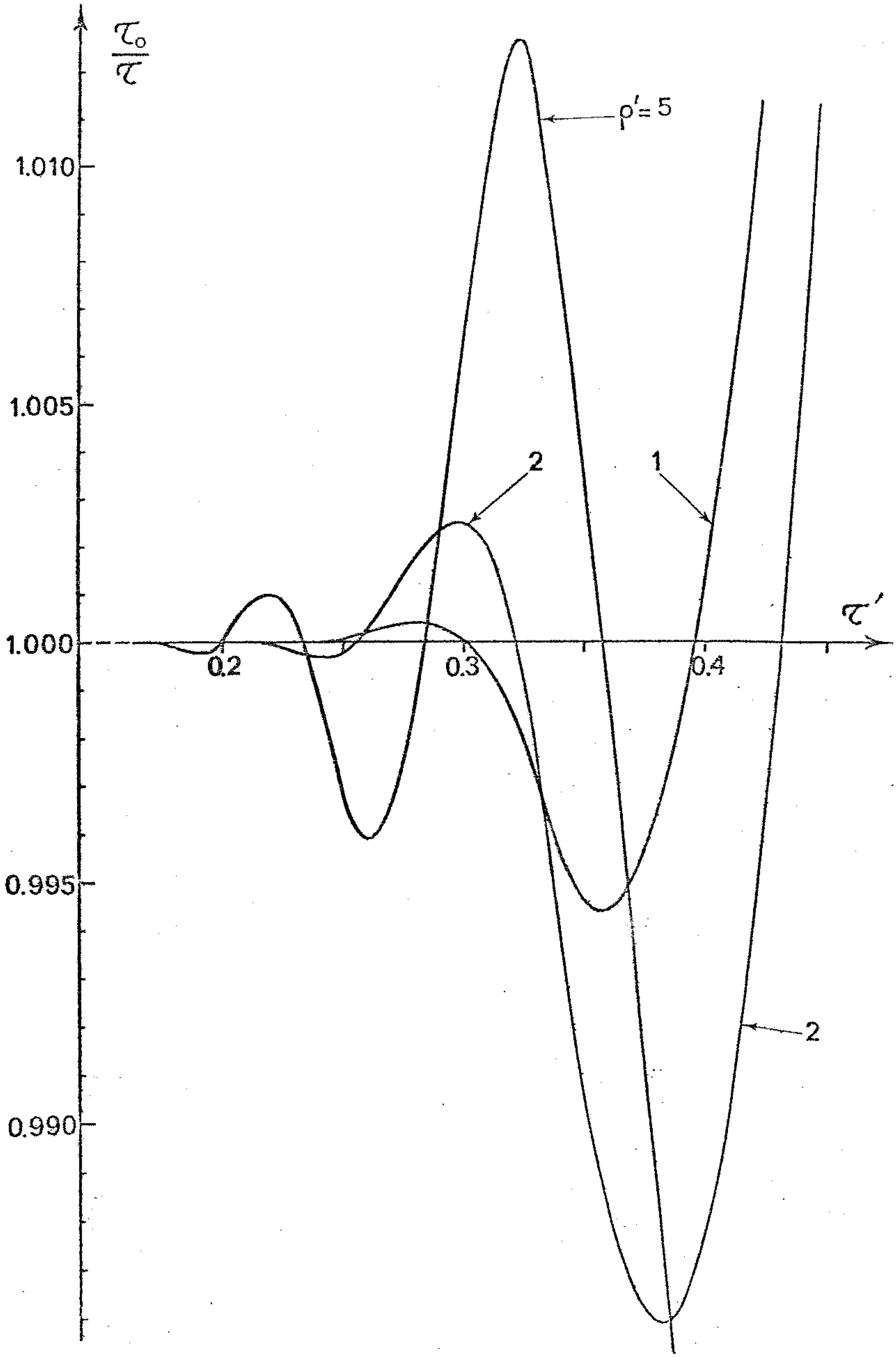

Figure 1 - Graphical représentation of the ratio τ_o/τ, for $\tau' < 0.4$ and some values of ρ'. This plot permits to determine the accuracy of the approximate formula.
(Reproduced from [2]).